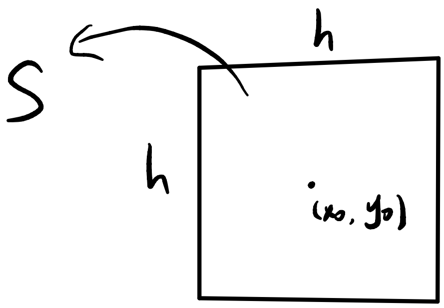
Derivation of the heat equation

Consider an infinite metal plane $\mathbb{R}^2$.

Let the temperature at the point (x, y) at time t be denoted by $u(x, y, t)$.

Consider a small square centered at (x_0, y_0) of side length h.



The amount of heat energy in S at time t is

$$H(t) = \sigma \iint_S u(x, y, t)\,dx\,dy$$

where $\sigma > 0$ is a constant called the specific heat of the material.

The heat flow into S is

$$\frac{\partial H}{\partial t} = \sigma \iint_S \frac{\partial u}{\partial t}\,dx\,dy \simeq \sigma h^2 \frac{\partial u}{\partial t}(x_0, y_0, t)$$

Newton's Law of Cooling

Heat flows from the higher to the lower temperature at a rate proportional to the difference, i.e., the gradient.

Therefore, the heat flow through the side on the right is

$$-kh\frac{\partial u}{\partial x}(x_0+\frac{h}{2}, y_0, t)$$

where $k>0$ is the conductivity of the material.

Thus, the heat flow into S is given by

$$kh\Big[\frac{\partial u}{\partial x}(x_0+\frac{h}{2}, y_0, t)-\frac{\partial u}{\partial x}(x_0-\frac{h}{2}, y_0, t)$$
$$+\frac{\partial u}{\partial y}(x_0, y_0+\frac{h}{2}, t)-\frac{\partial u}{\partial y}(x_0, y_0-\frac{h}{2}, t)\Big]$$

By Mean Value Theorem,

$$\frac{\sigma}{k}\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$$

$\square$

<u>Review of ODE</u>

(I) $y'+ay=0$

Sol: $\frac{dy}{dt}=-ay$

$\Leftrightarrow \frac{dy}{y}=-a\,dt$

$\Leftrightarrow d(\log y)=d(-at)$

$\Leftrightarrow \log y=-at+C_1$

$\Leftrightarrow y=C_2e^{-at}$ where $C_2=e^{C_1}$

$\square$

(II) $y'' + ay = 0$

Sol: $y'' + ay = 0$

$$\Leftrightarrow \quad y'' + \sqrt{-a}\,y' - \sqrt{-a}\,y' - (-a)y = 0$$

$$\Leftrightarrow \quad (y' + \sqrt{-a}\,y)' = \sqrt{-a}\,(y' + \sqrt{-a}\,y)$$

$z = y' + \sqrt{-a}\,y$

$$\Leftrightarrow \quad z' = \sqrt{-a}\,z$$

$$\Leftrightarrow \quad z = C_1 e^{\sqrt{-a}t}$$

$$\Leftrightarrow \quad y' + \sqrt{-a}\,y = C_1 e^{\sqrt{-a}t}$$

Similarly, one can show

$$y' - \sqrt{-a}\,y = C_2 e^{-\sqrt{-a}t}$$

Therefore,

$$2\sqrt{-a}\,y = C_1 e^{\sqrt{-a}t} - C_2 e^{-\sqrt{-a}t}.$$

Hence,

$$y = C_3 e^{\sqrt{-a}t} + C_4 e^{-\sqrt{-a}t}$$

where $C_3 = \frac{C_1}{2\sqrt{-a}}$ and $C_4 = -\frac{C_2}{2\sqrt{-a}}$.

$\square$